Vector Calculus Summary

Line integrals

- Over C of a scalar function (scalar field) f:

$$\int_C f(x,y)\,ds = \int_a^b f\big(x(t),y(t)\big)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_a^b f\big(\mathbf{r}(t)\big)|\mathbf{r}'(t)|\,dt$$

Or

$$\int_C f(x,y,z)\,ds = \int_a^b f\big(x(t),y(t),z(t)\big)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \int_a^b f\big(\mathbf{r}(t)\big)|\mathbf{r}'(t)|\,dt$$

- Over C of a vector field

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\big(\mathbf{r}(t)\big)\cdot\mathbf{r}'(t)\,dt = \int_C P\,dx + Q\,dy \quad \text{OR} \quad \int_C P\,dx + Q\,dy + R\,dz$$

$$\left(\text{These really mean } \int_C P\,dx + \int_C Q\,dy \;\text{ and }\; \int_C P\,dx + \int_C Q\,dy + \int_C R\,dz\right)$$

Note: We generally parameterize these.

Fundamental Theorem for Line Integrals

$$\int_C \nabla f\cdot d\mathbf{r} = f\big(\mathbf{r}(b)\big) - f\big(\mathbf{r}(a)\big), \text{where } \mathbf{r}(t),\;\; a\le t\le b \;\text{ describes } C$$

$$= f(x_2,y_2) - f(x_1,y_1) \;\;\text{or}\;\; = f(x_2,y_2,z_2) - f(x_1,y_1,z_1)$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = 0 \text{ for all closed paths } C \;\Leftrightarrow\; \int_C \mathbf{F}\cdot d\mathbf{r} \;\text{ is independendent of path}$$

$$\Longrightarrow\; \mathbf{F} \text{ is a conservative vector field}$$

$$\Longrightarrow\; \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

The last implication becomes if and only if and only if (⟺) if the partial derivatives are continuous throughout an open, simply connected region *D*, the domain of the vector field **F.**

If $\mathbf{F}$ is conservative, then $\mathbf{F} = \nabla f$ for some potential function f, and we can use the Fundamental Theorem for Line Integrals. If $\mathbf{F}$ is not conservative, we use the parameterized form given above $\left(\int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt\right)$, which becomes

$\iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$ if C is the boundary of the closed region D, by Green's Theorem.

(Note: Green's Theorem is stated below.)

To determine whether or not $\boldsymbol{F}$ is conservative (that is, whether or not to use the Fundamental Theorem for Line Integrals), in $\mathbb{R}^2$, check if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

In $\mathbb{R}^3$, check if $\text{curl}\, \mathbf{F} = \nabla \times \mathbf{F} = \mathbf{0}$. If we find that $\mathbf{F}$ is conservative, we find the potential function f by integrating:

$$f = \int P\, dx, \qquad f = \int Q\, dy, \qquad \text{(and in } \mathbb{R}^3\text{)}, \quad f = \int R\, dz$$

<u>Green's Theorem</u>

$$\int\limits_C \mathbf{F} \cdot d\mathbf{r} = \oint\limits_C P\, dx + Q\, dy = \int\limits_{\partial D} P\, dx + Q\, dy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Notes: $C = \partial D$ is the simple, positively oriented boundary curve of D. The symbol $\oint_C$ is used to indicate positive orientation.

<u>Area of a parametric surface</u>

$A(S) = \iint\limits_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$, where u and v are parameters

If x and y are the parameters, we have

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Surface integrals

- Of a scalar field $f(x, y, z)$:

$$\iint\limits_S f(x, y, z)\, dS = \iint\limits_D f\big(\mathbf{r}(u, v)\big)\, |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

Note that $dS = |\mathbf{r}_u \times \mathbf{r}_v|\, dA$.

- Of a vector field $\mathbf{F}(x, y, z)$:

$$\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = \iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint\limits_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA$$

Note: $d\mathbf{S} = \mathbf{n}\, dS = |\mathbf{r}_u \times \mathbf{r}_v|\, dA$, where $\mathbf{n}$ is a unit normal vector to the surface S and $|\mathbf{r}_u \times \mathbf{r}_v|$ is a normal vector to S.

If x and y are the parameters, we have

$$\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = \iint\limits_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA,$$

for upward orientation. The signs of the integrand change for downward orientation.

Stokes' Theorem

$$\int\limits_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S},$$

where C is the positively oriented piecewise-smooth boundary curve of S, an oriented piecewise-smooth surface.

The Divergence Theorem

$$\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = \iiint\limits_E \operatorname{div} \mathbf{F}\, dV,$$

where S is the boundary surface of E, a solid region whose surfaces are continuous, with outward orientation.